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PARAMEŚVARA'S RULE FOR THE CIRCUMRADIUS OF A CYCLIC QUADRILATERAL

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Summaries

The expression for the circumradius of a cyclic quadrilateral in terms of its sides, usually attributed to L'Huilier in 1782, was known in India to Parameśvara (circa 1430). The present paper contains the original Sanskrit text of the rule, its English translation, and a discussion of its derivation as given by Śaṅkara Vāriar in his Kriyākramakarī *(16th Century) along with relevant historical remarks.*

चक्रीय चतुर्भुज के भुजमानों से उसके परिगत वृत्त की त्रिज्या के ज्ञान एवं प्रकाशन (१७८२ ई.) का श्रेय एक यूरोपीय लेखक को दिया गया है। लेकिन तत्संबंधी नियम भारत में परमेश्वर (लगभग १४३० ई.) को पहले से ही ज्ञात था। निम्न लेख में उस नियम का मूल संस्कृत पाठ, आंग्ल अनुवाद, तथा शंकर कृत क्रियाक्रमकरी (१६ वीं शताब्दी) में वर्णित उपपत्ति का विवेचन सुसंगत ऐतिहासिक टिप्पणियों के साथ दिये हैं।

Let $ABCD$ be a cyclic quadrilateral with sides AB, BC, CD, and DA equal to a, b, c, and d respectively. Smith [1958, II, 287] stated that S. A. J. L'Huilier discovered and published in 1782 a formula which reduces

$$(1) \qquad R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}} \quad \ldots$$

where R is the radius of the circle circumscribing the quadrilateral and s is its semi-perimeter. The formula (1) was already known about 350 years earlier in India and is given verbally by Parameśvara (circa 1360-1455) in his commentary (before 1432) on the *Līlāvatī* (circa 1150) of Bhāskara II

[Saraswathi 1969, 69; Sarma 1972, 19].

The purpose of the present paper is to bring to the notice of scholars the Sanskrit verse and the Indian derivation of the rule as found in another commentary, called *Kriyākramakarī* (16th century), on the *Līlāvatī* recently published [Sarma (editor) 1975].

The composition of the *Kriyākramakarī* (=KKK) commentary was started by Saṅkara Vāriar (c. 1500-1560) and, after his death, finished by Mahiṣamaṅgalam Nārāyaṇa (1540-1610). The rule and its rationale are found in the portion (c. 1534) which was written by Saṅkara [Sarma (ed.) 1975, xxii].

The original Sanskrit text of the rule as found in the KKK (p. 363), and which is almost the same as that given by Parameśvara, is

दोष्णां द्वयोर्द्वयोर्घातयुतीनां तिसृणां वधे ।
एकैकोनेतरत्र्यैक्यचतुष्केण विभाजिते ॥
लब्धमूलेन यद् वृत्तं विष्कम्भार्धेन निर्मितम् ।
सर्वं चतुर्भुजं क्षेत्रं तस्मिन्नेवावतिष्ठते ॥

Doṣṇāṃ dvayordvayor-ghātayutīnāṃ tisṛṇāṃ vadhe/
Ekaikoṇetara-tryaikya-catuṣkena vibhājite
Labdhamūlena yad vṛttaṃ viṣkambhārdhena nirmitam/
Sarvaṃ caturbhujaṃ kṣetraṃ tasminnevāvatiṣṭhate

This may be translated almost literally thus:

"The three sums of the products of the sides taken two at a time are to be multiplied together and divided by the tetrad formed by diminishing one (of the sides) at a time from the sum of the other three. If a circle is drawn with the square-root of the quotient (just obtained) as semi-diameter, the whole quadrilateral figure will be located therein."

That is,

$$(2)\qquad R = \sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(b+c+d-a)(c+d+a-b)(d+a+b-c)(a+b+c-d)}} \quad \ldots$$

which is equivalent to (1).

After explaining the rule, the KKK gives (pp. 364-65) its rationale (*upapatti*), using the following three results.

Lemma I : The product of the flank sides of any triangle divided by the diameter of its circumscribed circle is equal to the altitude of the triangle.

Lemma II : The area of the cyclic quadrilateral is given by

$$(3)\qquad S = \sqrt{(s-a)(s-b)(s-c)(s-d)} \quad \ldots$$

Lemma III: (cf. Ptolemy's Theorem): Let $ABCD'$ be the quadrilateral formed from $ABCD$ by interchanging the sides AD and CD, that is, by taking $AD' = CD = c$ and $CD' = AD = d$. If x,y,z, denote the three diagonals AC, BD, and BD' respectively, then $yz = ab + cd$, $zx = bc + da$, $xy = ca + bd$.

Lemma I was known to Indians for about a thousand years before the date of the KKK. In the equivalent form circumradius=(product of flank sides)/(twice the altitude), ... it is implied in a rule given by Brahmagupta (A.D. 628) in his *Brāhmasphuṭa-siddhānta*, XII, 27 [Sharma (ed.) 1966, III, 834; Gupta 1974 b, 173]. The KKK itself proves it separately (pp. 365-366). It is also used and proved in another Indian work called *Yuktibhāṣā* (=YB) which is attributed to Jyeṣṭhadeva (c. 1500-1610) [Sarma 1972, 59-60; Thampuran and Aiyar 1948, 231, 243-246].

Lemma II has been very popular in India since it was first stated by Brahmagupta in his *Brāhmasphuṭa-siddhānta* (=BSS), XII, 21 [Sharma (ed.) 1966, III, 816; Gupta 1974 a, 34-35]. According to Dr. K. S. Shukla (a great authority on Hindu astronomy and mathematics), Brahmagupta and other early Indian mathematicians have committed an error in declaring the formula (3) "as applicable to all quadrilaterals (with unequal altitudes), when in fact it is applicable to cyclic quadrilaterals only" [Shukla (ed.) 1959, p. 90 (translation)]. However, a recent scholar has been "unable to accept that Brahmagupta could have imagined that his rules would apply to all quadrilaterals whatsover" [Pottage 1974, 354]. The whole difficulty arises out of the fact that Brahmagupta himself has neither explicitly specified the correct range of application of his rule (3) nor given any derivation for it. But this state of affairs was not an unusual feature of ancient Indian mathematical texts.

Gaṇeśa in his commentary (A.D. 1545) on the *Līlāvatī* [Apte (ed.) 1937, 156-157] attempted to prove the rule (3) but the demonstration is incorrect [Inamdar 1946, 36-42].

A detailed proof of Lemma II is found in the YB (pp. 247-257). When the product of the two diagonals is needed in the course of this proof, it is derived by making use of the following (the so-called Brahmagupta's expressions for diagonals of a cyclic quadrilateral):

(4) $$x = \sqrt{(ac + bd)(ad + bc)/(ab + cd)} \ldots$$

(5) $$y = \sqrt{(ac + bd)(ab + cd)/(ad + bc)} \ldots$$

These results are given by Brahmagupta in his BSS, XII, 28 [Sharma (ed.) 1966, III, 836] and are considered to be the "most remarkable in Hindu Geometry and solitary in its excellence" by a recent historian of mathematics [Eves 1969, 187]. The formula (5) is stated to be rediscovered in Europe by W. Snell who gave it in his edition (1619) of Van Ceulen's work [Smith 1958, II, 287]. In fact the expressions (4) and (5) are separately derived in the YB (p. 233) from Lemma III which we now consider.

The Indian discussion of Lemma III is quite interesting because of the concept of the third diagonal of a cyclic quadrilateral. Bhāskara II had shown that the interchange of two adjacent sides of a (cyclic) quadrilateral alters the length of one of the diagonals (thereby getting a third diagonal) and this area and perimeter preserving construction appears in his *Līlāvatī* [Apte (ed.) 1937, II, 187; Colebrooke (tr.) 1967, 110; Pottage 1974, 306].

The geometry of the three diagonals of a cyclic quadrilateral is discussed in greater detail by Nārāyaṇa Paṇḍita (not to be confused with Mahiṣamaṅgalam Nārāyaṇa mentioned above) in his *Gaṇita-kaumudī* (c. 1356). For instance, Rule 52 from the *kṣetravyavahāra* portion of the work runs as follows [Dvivedi (ed.) 1942, 59]:

द्विगुणव्यासविभक्ते त्रिकर्णघातेऽथवा गणितम् ।
त्रिभुजे चतुर्भुजे वा व्यासस्य दलं प्रजायते हृदयम् ॥ ५२॥

Dviguṇa-vyāsa-vibhakte trikarṇa-ghāte'thavā gaṇitam/
Tribhuje caturbhuje vā vyāsasya-dalaṃ prajāyate hṛdayam//

"the product of the three diagonals divided by twice the diameter (of the circumscribed circle) is the area of a triangle or quadrilateral; half of the diameter becomes the *hṛdayam* (circumradius)."

That is, Area $S = x\ y\ z\ /\ 4R$ for a cyclic quadrilateral as well as a triangle (in which case the three sides themselves will be its three diagonals).

Rule 137½ from the same portion of the work gives the above relation in the form $R = x\ y\ z\ /\ 4$ (area). It is interesting to note that, after stating this rule, Nārāyaṇa criticized Brahmagupta's rule for the circumradius [BSS, XII; Pottage 1974, 334-335] as being *avyāpaka* ('not universal') and further said that Lalla (c. 748 A.D.) and Śrīpati (c. 1039) blindly followed Brahmagupta in this respect [Dvivedi (ed.) 1942, 175].

The discussion of the three diagonals as found in the KKK is more subtle. Firstly, it shows that in a cyclic quadrilateral more than three diagonals are not possible. The arguments given are substantially as follows (p. 351):

Let α, β, γ, δ be the angular measures of the arcs corresponding to the sides *a, b, c, d* (respectively) of a cyclic quadrilateral. Now a sum of any two arcs can be made to define a diagonal. Hence there can be six cases. But because $\alpha + \beta + \gamma + \delta = 360°$ there will be only three final possibilities (for example, if $\alpha + \beta$ defines one diagonal, $\gamma + \delta$ will define the same diagonal). Hence only three diagonals are possible (our x, y, z will be found to correspond to $\alpha + \beta$, $\beta + \gamma$, $\gamma + \alpha$ respectively).

The complete proof of Lemma III as given in the KKK (pp. 349-351) may be briefly mentioned in terms of modern symbols as follows.

Simple geometrical proofs of the following two preliminary results are given

$$ch^2\theta - ch^2\phi = ch(\theta+\phi).ch(\theta-\phi) \tag{6}$$

$$ch\lambda.ch\mu = ch^2\{(\lambda+\mu)/2\} - ch^2\{(\lambda-\mu)/2\}, \tag{7}$$

where ch stands for the chord of an arc (For the sine function also, similar results hold good). Now we have, with reference to the accompanying figure,

$$\begin{aligned} ab + cd &= ch\,\alpha.ch\,\beta + ch\,\gamma.ch\,\delta \\ &= ch^2\{(\alpha+\beta)/2\}-ch^2\{(\beta-\alpha)/2\}+ch^2\{(\gamma+\delta)/2\}-ch^2\{(\gamma-\delta)/2 \end{aligned} \tag{8}$$

by one of the above results.

If E and W be the mid-points of the arcs ABC and ADC respectively, then

$$ch\{(\alpha+\beta)/2\} = AE, \text{ and } ch\{(\gamma+\delta)/2\} = AW.$$

Also AEW is a right-angled triangle with hypotenuse EW equal to the diameter of the circle. Hence (8) gives

$$\begin{aligned} ab + cd &= (2R)^2.- ch^2\{(\beta-\alpha)/2\}-ch^2\{(\gamma-\delta)/2\} \\ &= ch^2(180^\circ - \tfrac{\beta-\alpha}{2}) - ch^2\{(\gamma-\delta)/2\} \\ &= ch(180^\circ - \tfrac{\beta-\alpha}{2} + \tfrac{\gamma-\delta}{2}).ch(180^\circ - \tfrac{\beta-\alpha}{2} - \tfrac{\gamma-\delta}{2}) \end{aligned}$$

by the formula (6). Because $\alpha + \beta + \gamma + \delta = 360^\circ$, we finally get

$$\begin{aligned} ab + cd &= ch(\alpha + \gamma).ch(\alpha + \delta) \\ &= (\text{chord of the arc } BAD')\cdot(\text{chord of the arc } BAD) \\ &= BD' \cdot BD = z\cdot y \end{aligned}$$

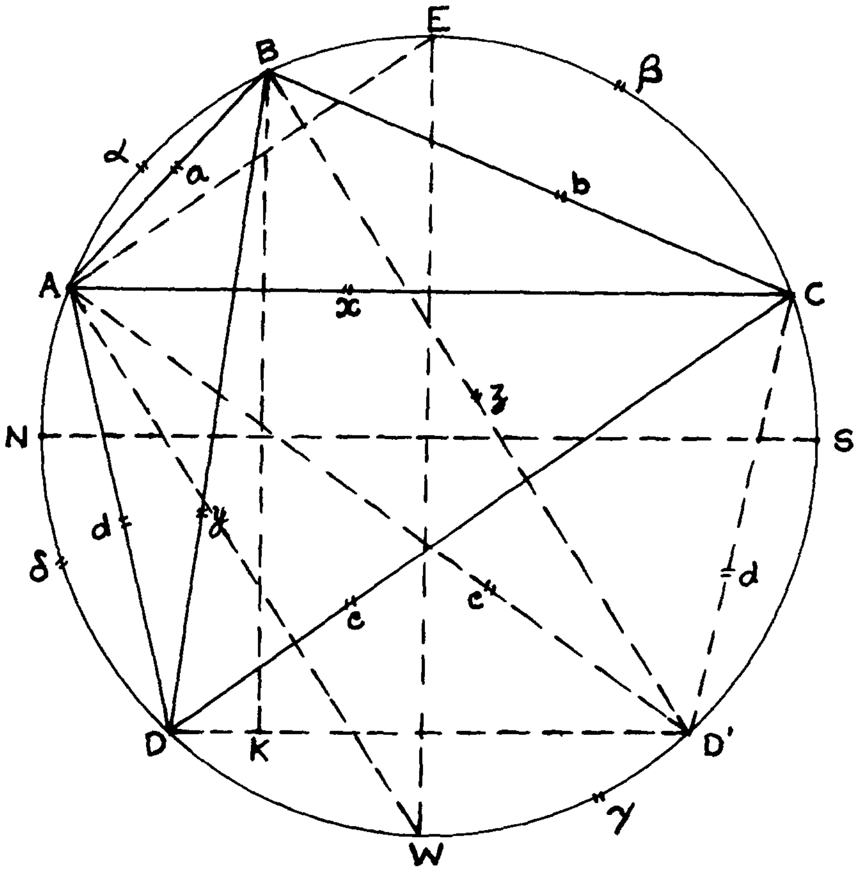
which is the first equation of Lemma III. The other equations can be derived similarly, and the proof of the Lemma III is thus completed. The proof given in the YB (pp. 228-233) is somewhat similar to this.

These Indian proofs of the so-called Ptolemy's Theorem are radically different from that given about 1500 years earlier by Ptolemy in his *Almagest* [Taliaferro 1952, 16-17].

After proving Lemma III, the KKK (p. 351) derives the expressions for the squares of the two diagonals (x and y) from it, that is, from the equations in Lemma III. These expressions are equivalent to the famous Indian formulas (4) and (5). Finally, a similar expression for the third diagonal is also derived but "it is not given here (that is, in the original text) because of its non-utility *(anupayoga)*", the KKK says. Almost the same discussion is found in the YB (p. 233). These

Indian derivations may be contrasted with the conjectural Brahmaguptan proofs as suggested by Pottage [1974, 344-349].

The derivation of the main result (1) may now be presented briefly.



The KKK starts (p. 364) by asking us to draw a diagram similar to the accompanying figure. In it *EW* and *NS* are east-west and north-south lines (east was represented upwards by Indians). *BK* is drawn perpendicular to *DD'* (which is parallel to *AC*). Other details are self evident in the figure.

By Lemma I, applied to the triangle *BDD'*, we get

(9) $$\text{perp. } BK = yz/2R \ldots$$

This perpendicular *BK* will be the sum of the altitudes of the two triangles *BAC* and *DAC* into which the quadrilateral *ABCD* is divided by the diagonal *AC* (which becomes their common base). Thus, the area of the quadrilateral

(10) $$S = (1/2)BK.x \ldots$$

Therefore, by (9) and (10), we get

(11) $$R = xyz/4S \ldots$$

As stated above, this result was already known to Nārāyaṇa Paṇḍita (c. 1356).

Parameśvara's rule (1) now immediately follows from (11) by using Lemma II and Lemma III, that is, by multiplying the equations in Lemma III to get xyz as needed in (19).

Just after completing the proof, the KKK (p. 365) adds an intelligent remark which renders unnecessary the alternate reading (involving the word *vadha* or *ghāta*, that is, 'product') of the original Sanskrit stanza, as mentioned by the editor and found quoted elsewhere [Saraswathi 1969, 69; Sarma 1972, 19].

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